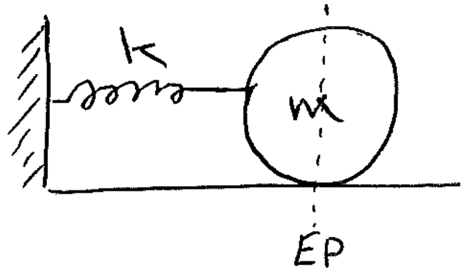
Classical & Quantum Waves

Lecture 2-1

Simple harmonic oscillator (continued)

- Last time we considered SHO of a mass attached to a spring:



- Using Hooke's law, $F = -kx$, we arrived at the differential eq.

$$m \frac{d^2x}{dt^2} = -kx \longrightarrow \boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

with $\omega^2 \equiv \frac{k}{m}$

- We want to know the solution to this eq., $x(t)$

Displacement, velocity & acceleration in SHO

- Physical intuition (and/or experiments) suggests that a cosine function describes SHO:

Guess that $x(t) = A \cos\Big(\underbrace{2\pi \frac{t}{T}}\Big)$

Phase angle goes from $0 \rightarrow 2\pi$ over time T

$x = A$ at $t = 0$

• Is this a solution to our differential eq.

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad ?$$

• Define $\boxed{\omega \equiv \frac{2\pi}{T}} = 2\pi\nu$ angular frequency: how many 2π ~~cycl~~ cycles per sec

~~unit:~~ unit: $[\omega] = \frac{\text{rad}}{\text{s}}$

→ $x = A\cos\omega t$

displacement

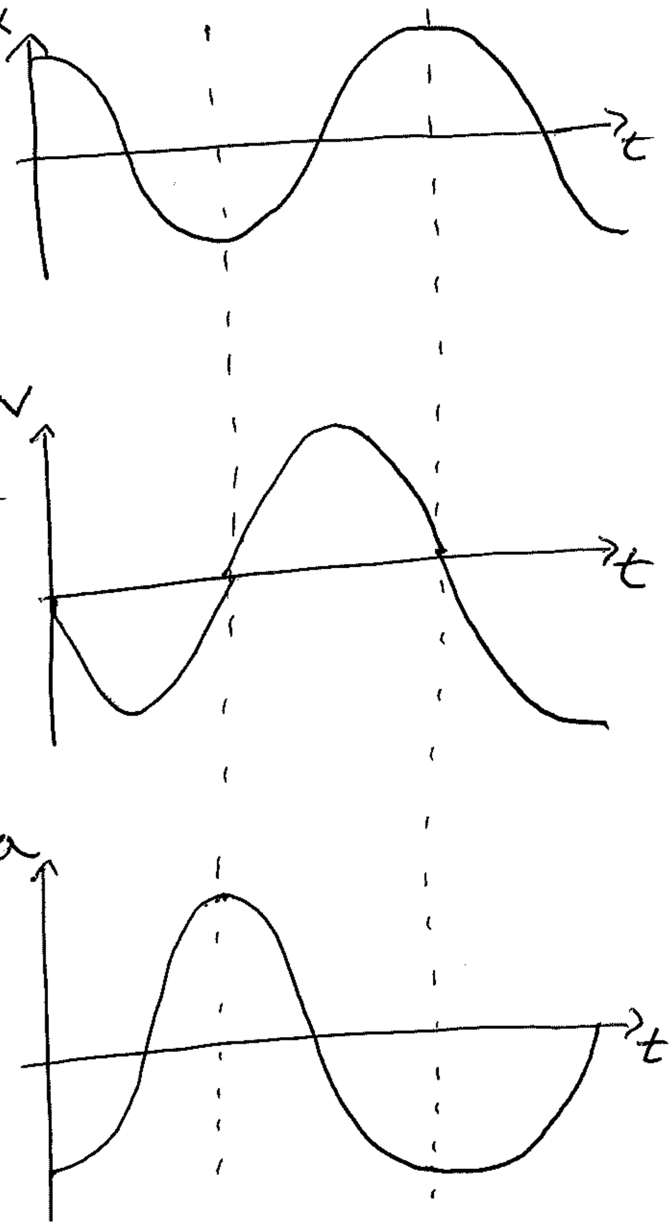
→ $\frac{dx}{dt} = v = -\omega A \sin\omega t$

velocity

→ $\frac{d^2x}{dt^2} = a = -\omega^2 A \cos\omega t$

$= -\omega^2 x$

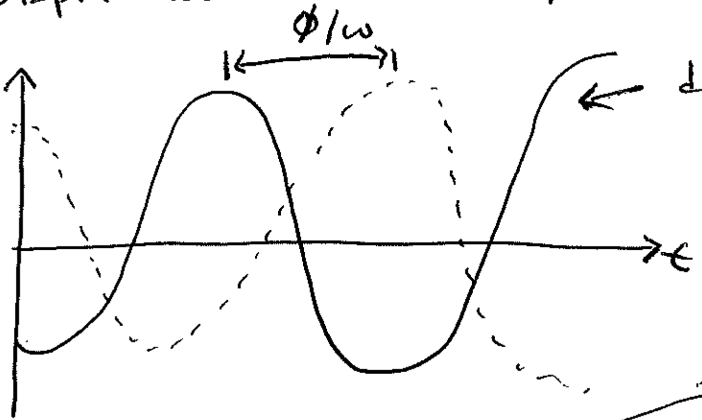
acceleration



Force $\propto$ a is max. at turning points

What is the general solution for SHO?

Displacement and velocity at $t=0$ can be arbitrary!



← displaced to the left by a time ϕ/ω

$$\rightarrow \boxed{x(t) = A\cos(\omega t + \phi)}$$

ϕ is called <u>phase angle</u>

- General solution of $\dfrac{d^2x}{dt^2} = -\omega^2 x$

- Alternative form: $\boxed{x(t) = a\cos\omega t + b\sin\omega t}$ (can show equivalent to $A\cos(\omega t + \phi)$)

- Note: ω is determined by properties of the oscillator

  → "natural frequency of oscillation", e.g. $\omega \equiv \sqrt{\dfrac{k}{m}}$ for mass on spring

  - Oscillation frequency independent of how large initial amplitude

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Energy of a SHO

- Working with energy is powerful and simpler to calculate

- Mass on a spring has:
  - kinetic energy $K = \frac{1}{2} m v^2$
  - potential energy $U = \int_0^x kx'\,dx' = \frac{1}{2} k x^2$

(Work to extend spring from x' to $x'+dx'$ is $kx'dx'$)

- Energy is always conserved, so total energy is

$$\boxed{E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2}$$

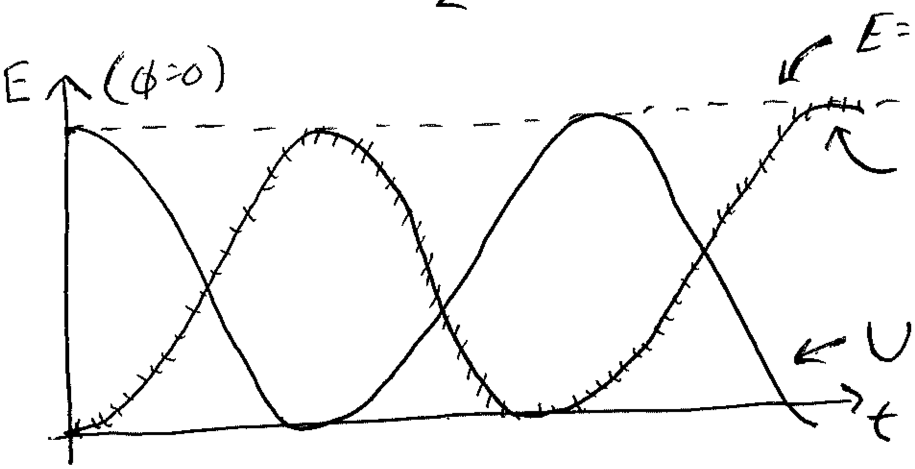
- Using general solution $x(t) = A\cos(\omega t + \phi)$, $\frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \underset{\omega^2 = \frac{k}{m}}{=} \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

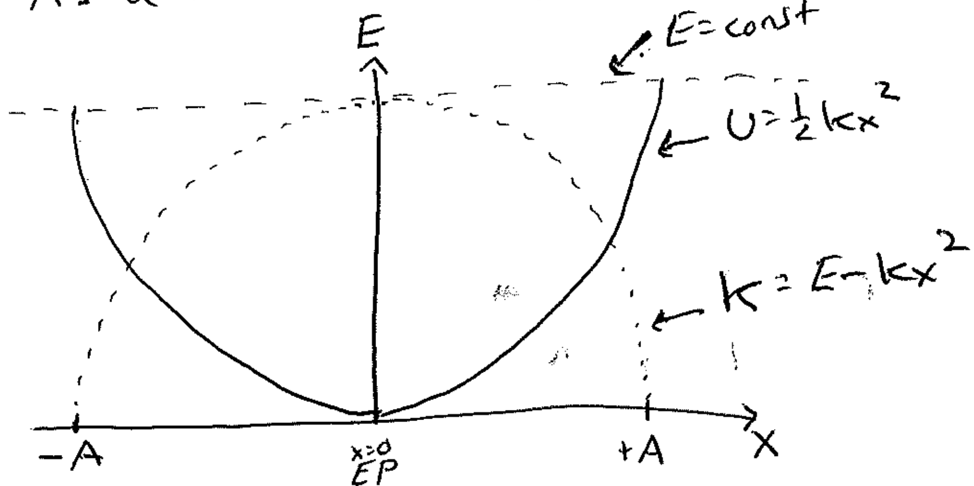
$$\rightarrow E = K + U = \frac{1}{2}kA^2\Big(\underbrace{\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)}_{=1}\Big)$$

$$\rightarrow E = \frac{1}{2}kA^2$$



Energy flows b/t K and U!

- As a function of displacement:



- Mass is confined in parabolic potential well
  - ↳ Always gives SHO!

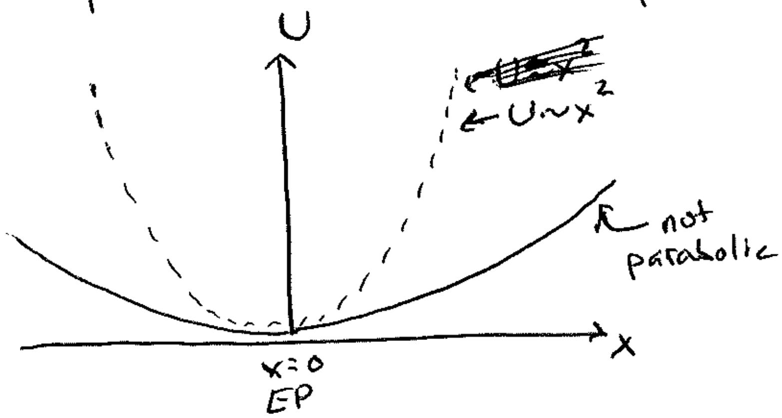
(e.g., marble in a bowl)

What if potential well not parabolic?

- Almost all potential wells can be approximated as parabolic close to EP

↳ why most oscillating systems exhibit SHO when ~~close~~ displacement is small

Explanation: Assume non parabolic potential



- Taylor's theorem: A function $f(x)$ can be expanded around $x=a$

$$f(x) = f(a) + \frac{(x-a)}{1!}\left(\frac{df}{dx}\right)_{x=a} + \frac{(x-a)^2}{2!}\left(\frac{d^2f}{dx^2}\right)_{x=a} + \dots$$

- Expand potential well around EP, $x=0$:

$$U(x) = \underbrace{U(0)}_{\substack{\text{Potential energy offset} \\ \equiv 0}} + \underbrace{x\left(\frac{dU}{dx}\right)_{x=0}}_{=0 \text{ b/c at minimum}} + \frac{x^2}{2}\left(\frac{d^2U}{dx^2}\right)_{x=0} + \dots$$

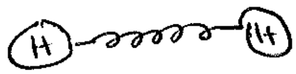
- for small x, $U(x) \simeq \frac{x^2}{2}\left(\frac{d^2U}{dx^2}\right)_{x=0}$

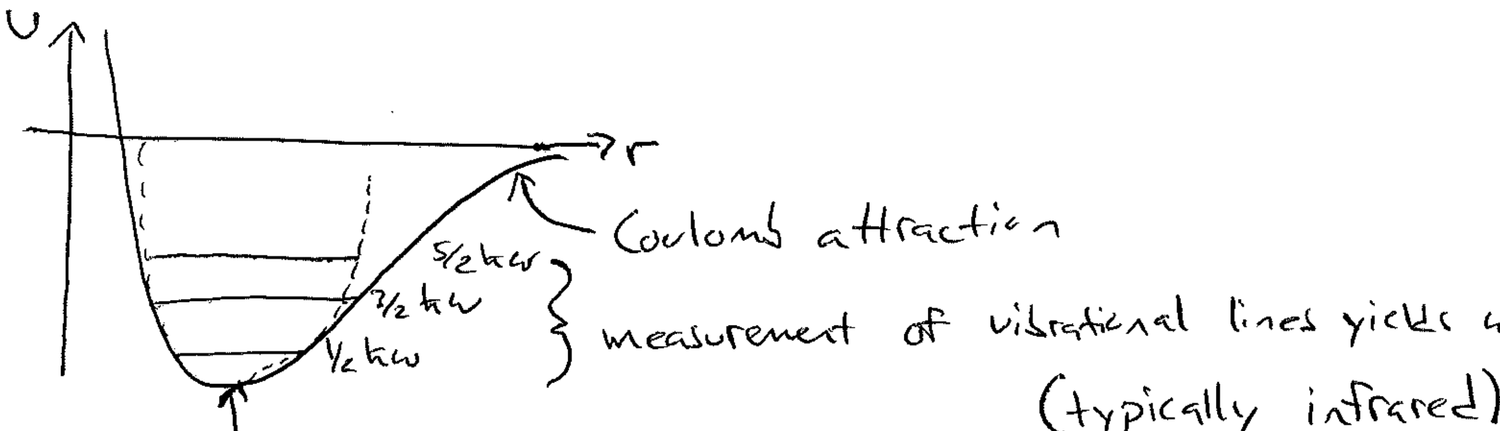
spring constant k for mass on spring

· Many examples of SHOs around us:

system	Period T [sec]
water in a lake	$10^2 - 10^4$
bridges / buildings	$1 - 10$
clock pendulum	1
instruments	$10^{-3} - 10^{-2}$
piezo crystal	10^{-6}
molecules	10^{-15}

· Hydrogen molecule H_2





EP $r_0 \sim 0.7 \times 10^{-10}$ m

→ measurement of molecule optical spectra gives info on bond strength ~~via ω^2 via $k = \frac{1}{2} m$~~

↳ get "spring constant" of molecule

(see worked example p. 16-17 King)